Aggregation of Discount Rates: an Equilibrium Approach

Elyès Jouini, Clotilde Napp

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Abstract

We address the problem of a social planner who, as in Weitzman (2001), gathers data on individual discount rates and wants to infer the socially efficient discount rate. We propose an equilibrium approach, that relies on the techniques of Jouini et al. (see Jouini, E., Marin J.-M., and C. Napp, 2009. Discounting and Divergence of Opinion, to appear, *Journal of Economic Theory*). We provide an aggregation formula and analyse the properties of the resulting discount rate. We compare the expression we obtain with those previously proposed in the literature. We analyse the impact of shifts in the distribution of individual discount rates. Finally, we apply our approach to Weitzman (2001)'s data to propose discount rates for public sector Cost-Benefit Analysis, in particular for the long term.

1. Introduction

The appropriate social discount rate to apply in public sector cost-benefit analysis is a contentious issue. This is especially true for long term projects, for which financial markets cannot provide any guideline. As Weitzman (2001, p.261) states "There does not now exist, nor has ever existed, anything remotely resembling a consensus, even -or, perhaps one should say especially- among the 'experts' on this subject".

In this note we address the problem of a social planner who, as in Weitzman (2001), has consulted a group of agents about the discount rate to apply for costs or benefits occurring at a given date t and wants to aggregate the proposed rates into a socially efficient discount rate.

As underlined by e.g. Nordhaus (2007) or Weitzman (2007), there is an important distinction between the *utility* social discount rate and the *consumption* social discount rate.

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The former refers to a pure time preference rate that discounts utility. The latter is the rate used to discount future consumption; it is determined by the time preference rate, but also by the anticipations about the future of the economy. The (extended) Ramsey equation[1] illustrates the distinction and the relation between the two rates. In this note, the rates proposed by the agents, as well as the socially efficient discount rate to be inferred, are consumption discount rates, since they are to be applied to cost-benefit analysis. This is also the case in Weitzman (2001); indeed, as Weitzman makes it clear in his questionnaire : "What I am here after is the relevant interest rate for discounting real-dollar changes in future goods and services –as opposed to the rate of pure time preference on utility"[2].

Weitzman (1998, 2001) deal with this problem by adopting a certainty equivalent approach. In this approach, the social discount factor is taken to be the (arithmetic) average of the discount factors proposed by the agents in the panel. Gollier (2004) considers as arbitrary approaches that do not rely on realistic preferences of human beings towards risk and time and suggests that an equilibrium analysis is maybe the cost to be paid to make policy recommendations that have an economic sense.

We propose an approach that relies on an equilibrium analysis. We consider that each proposed rate corresponds to a specific calibration of Ramsey formula and reflects then, for each individual, her pure time preference rate (or her conception of intergenerational equity) ρ_i and her beliefs about the future growth of the economy (μ_i, σ_i). The divergence in the proposed individual discount rates R^i stems then from divergence in individual tastes and beliefs. We consider that the participants to the survey are representative in the sense that each of them represents the tastes and beliefs of the same portion of the population. It is then natural to adopt as the socially efficient discount rate the equilibrium discount rate in the economy made of agents with the heterogeneous beliefs and tastes of the participants to the survey. For example, if the panel is made of three participants proposing discount rates $(R^i)_{i=1,2,3}$, corresponding to characteristics $(\rho_i, \mu_i, \sigma_i)_{i=1,2,3}$, then we take as the socially efficient discount rate the equilibrium discount rate in an economy made of one third of agents with each set of characteristics. This means that we have transformed the problem of aggregating data on heterogeneous discount rates into the problem of aggregating data on heterogeneous beliefs and tastes. We can then apply the techniques[3] of Jouini et al. (2009) in order to obtain the expression of the socially efficient discount rate.

We compare our expression for the socially efficient discount rate with other expressions previously considered in the literature. In particular, our formula are different from Weitzman (1998). The equilibrium discount rate coincides with Weitzman (1998, 2001) certainty equivalent discount rate when all agents have the same pure time preference rate. In a more general setting, the discount rates of the more impatient agents are granted a higher weight.

[1]Letting R denote the consumption discount rate, and ρ the utility discount rate, the extended Ramsey formula gives the relation $R = \rho + \eta\mu - \frac{1}{2}\eta\,(1+\eta)\,\sigma^2$, where μ is the growth rate of the economy and η is the elasticity of marginal utility. The original Ramsey equation (Ramsey, 1928) was derived in a deterministic setting ($\sigma = 0$) and is given by $R = \rho + \eta\mu$. The extended Ramsey equation corresponds to a direct generalization in a stochastic setting. For the sake of completeness, we rederive it in the Appendix.

[2]Moreover, the fact that the given rates are on average equal to 4% confirms that the participants to the panel actually gave their discount rate for consumption.

[3]Jouini et al. (2008) deals with the determination of the equilibrium discount rate in an economy in which agents have heterogeneous beliefs and heterogeneous time preference rates.

A possible interpretation is as follows. When considering its intertemporal rate of substitution, the group must weigh more the agents with a higher shadow price of the intertemporal budget constraint, i.e. the more impatient members of the group. The overweighting of impatient agents discount rates implies that when tastes and beliefs are independent, the equilibrium discount rate is higher than the certainty equivalent discount rate for all horizons.

We determine the explicit expression of the socially efficient discount rate for specific distributions of the agents discount rates. We calibrate the model on Weitzman (2001)'s data. Our results suggest using the following approximation of within-period marginal discount rates for long term public projects: Immediate Future about 5 per cent; Near Future about 4 percent; Medium Future about 3 percent; Distant Future about 1.5 per cent and Far-Distant Future about 0 per cent. Except for the Far-Distant Future, these rates are slightly higher than those obtained by Weitzman (2001).

Finally, we determine which concepts of stochastic dominance on the distributions of individual discount rates lead to a clear impact on the equilibrium discount rate. We analyze the impact of standard shifts, like first or second stochastic dominance shifts as well as monotone likelihood ratio dominance shifts. Roughly speaking, more pessimism, more patience, more doubt as well as more heterogeneity in individual discount rates reduce the equilibrium discount rate.

Note that our approach also permits to aggregate utility discount rates[4] (pure time discount rates). It suffices to consider the specific case where there is no beliefs heterogeneity. We obtain that the equilibrium utility discount rate is a weighted average of the individual ones. Our formulas are then analogous to those of Lengwiler (2005). They coincide with those of Nocetti et al. (2008) and Gollier and Zeckhauser (2005) only for specific choices of Pareto weights. They differ from those in Reinschmidt (2002), in the same way as our formulas for the consumption discount rate differ from those in Weitzman (1998). We emphasize that, while these papers aim at aggregating individual *utility* discount rates, the aim of the present note is to aggregate individual *consumption* discount rates and to do it through an *equilibrium* approach.

All the agents are assumed to have logarithmic utility functions. The first reason is the central role of logarithmic utility functions. Jouini et al. (2009) shows that the log-utility setting is central in the analysis of beliefs heterogeneity: some biases are induced when dealing with power utility function $\frac{c^{1-\eta}}{1-\eta}$ with $\eta \neq 1$, these biases being in opposite directions depending on the position of η with respect to 1. The second reason is analytical tractability. Logarithmic utility functions lead to closed form formulas and it is easier to interpret in such a framework the impact of the different sources of heterogeneity on the equilibrium discount rates. However, the same methodology can be applied to more general utility functions but would require a numerical treatment instead of leading to closed form formulas.

All proofs are in the Appendix.

[4]The problem of aggregating individual utility discount rates has been studied by, among others, Reinschmidt (2002) through a certainty equivalent approach, Nocetti et al. (2008) through a Benthamite approach, Gollier-Zeckhauser (2005) through a Pareto optimality approach and Lengwiler (2005) through an equilibrium approach.

2. Equilibrium discount rate

Let us consider n individuals, who propose different discount rates (R^i) for cost-benefit analysis of public projects as in Weitzman (2001).

We assume that the discount rate proposed by agent i for costs or benefits occurring at date t corresponds to a specific parametrization of the extended Ramsey equation, i.e. the consumption discount rate R^i proposed by agent i is given by

$$R^i = \rho_i + \mu_i - \sigma_i^2$$

where $\rho_i > 0$, μ_i and σ_i^2 are respectively her pure time preference rate as well as the mean and the variance (by unit of time) of her distribution of the growth rate of aggregate consumption. The rate R^i is the equilibrium discount rate that would prevail if the economy was made of group i agents only. The divergence on the discount rates (R^i) results then from divergence on the parameters $(\rho_i, \mu_i, \sigma_i^2)$.

Assuming that agents differ in their expectation about the growth rate is fairly natural. Indeed, the expected growth rate reflects the opinion about the future. It suffices to look at experts forecasts to realise that there is no consensus about the future of the economy. It is also natural to assume that agents differ in their pure time preference rate since it may reflect their point of view about intergenerational equity as well as one's level of impatience. As an illustration of the possible divergence, the UK government recommends[5] a discount rate of 3.5% based on the following figures $\rho = 1\%$, $\mu - \sigma^2 = 2.5\%$ while the Stern review proposes a discount rate of 1.4% using $\rho = 0.1\%$ and $\mu - \sigma^2 = 1.3\%$. Note that both of them use logarithmic utility functions.

The problem now is to determine how to aggregate these discount rates into a consensus discount rate. We consider that the panel tastes and beliefs reflect those of the population. We shall then consider a complete markets economy with heterogeneous agents endowed with the same beliefs and tastes as those in the panel and we shall adopt the equilibrium discount rate in this economy as our consensus discount rate.

To summarise, we have

- N groups of agents,
- $w_i \equiv$relative size of group i,
- $\rho_i \equiv$ pure time preference rate of the agents in group i ,
- $t \equiv$ the time at which a cost or benefit is incurred, relative to the present time,
- a date 0 total consumption e_0 normalized to 1,
- $\ln \mathcal{N}((\mu_i - \frac{1}{2}\sigma_i^2)t, \sigma_i^2 t) \equiv$ group i's anticipated distribution[6] of aggregate consumption e_t at date t,

[5] In the HM Treasury's Green Book (2003).

[6] This is the case for instance if aggregate consumption is a geometric Brownian motion with drift μ_i and volatility σ_i. We will in Section 4 consider a more general setting with general distributions for aggregate consumption. The formulas are then less easy to handle.

- log utility functions,
- $R^i \equiv \rho_i + \mu_i - \sigma_i^2$, the individual discount rate for agents in group i.

The weights w_i model then the distribution of agents' characteristics in the economy, which also correspond by construction to the distribution of the characteristics of the agents in the panel.

2.1. Expression of the equilibrium discount rate and properties

Let us denote by P^i the probability over the states of the world that is considered by agent i. Under P^i, the distribution of aggregate consumption at date t is then given by $\ln\mathcal{N}((\mu_i - \frac{1}{2}\sigma_i^2)t, \sigma_i^2 t)$. We let M^i denote the density of P^i with respect to a given reference probability P. An Arrow-Debreu equilibrium is defined by a state price q_t^* and optimal consumption plans $\left(y^{*^i}\right)_{i=1,\ldots,N}$ such that each group maximizes its aggregate utility $\int_0^\infty \exp(-\rho_i t) E\left[M_t^i \log y_t^i\right] dt$ under its budget constraint $\int_0^\infty E^P\left[q_t y_t^i\right] dt \leq w_i \int_0^\infty E^P\left[q_t e_t\right] dt$ and markets clear i.e., $\sum_{i=1}^N y^{*^i} = e$.

We denote by A_t the equilibrium discount factor for horizon t, i.e. the price at date 0 of \$1 at date t. We denote by $R_t \equiv -\frac{1}{t}\ln A_t$ the discount rate for horizon t, i.e. the rate which if applied constantly for all intervening years would yield the discount factor A_t. We denote by $r_t \equiv -\frac{A_t'}{A_t}$ the marginal discount rate for horizon t, i.e. the rate of change of the discount factor. We have $R_t = \frac{1}{t}\int_0^t r_s ds$. Marginal and average rates of discount coincide when the discount rate is constant. In particular, for all i, the individual marginal discount rate r^i coincides with the individual discount rate R^i. However, the distinction between the two notions of discount rates can become important when the discount rate is time dependent (Groom et al., 2005).

We easily get, as in Jouini et al. (2009, Proposition 5.1), that the discount rate R_t is given by

$$R_t \equiv -\frac{1}{t}\ln\sum_{i=1}^N \frac{w_i\rho_i}{\sum_{j=1}^N w_j\rho_j}\exp\left(-R^i t\right), \tag{2.1}$$

that it decreases with t and that it converges to the lowest proposed rate $R_\infty = \inf_i R^i$. Note that our setting is slightly different from the setting in Jouini et al. (2009). Indeed, in Jouini et al. (2009), aggregate consumption follows a specific diffusion process. However, the proofs remain essentially the same. For the sake of completeness, we provide the proof of Equation (2.1) at the beginning of the Appendix. Moreover, we will show in Section 4 that the formula for R_t in Equation (2.1) remains valid in a very general Arrow Debreu setting, with continuous or discrete time, a finite number or a continuum of agents, and general distributions for aggregate consumption[7].

Analogously, we easily obtain the following results on the marginal discount rate.

[7] In a more general setting, the individual discount rates R_t^i are not given by the Ramsey formula and may then be dependent upon t.

Proposition 2.1. *1. The equilibrium marginal discount rate is given by*

$$r_t \equiv \sum_{i=1}^{N} \frac{w_i \rho_i \exp\left(-r^i t\right)}{\sum_{j=1}^{N} w_j \rho_j \exp\left(-r^j t\right)} r^i. \tag{2.2}$$

2. In the case of homogeneous beliefs ($\mu_i = \mu, \sigma_i = \sigma$), the equilibrium marginal discount rate is given by

$$r_t \equiv \sum_{i=1}^{N} \frac{w_i \rho_i \exp\left(-\rho_i t\right)}{\sum_{j=1}^{N} w_j \rho_j \exp\left(-\rho_j t\right)} \rho_i + \mu - \sigma^2. \tag{2.3}$$

3. The discount rate r_t decrease with t, and the asymptotic equilibrium discount rate is given by the lowest individual discount rate, i.e. $r_\infty = \inf_i r^i = R_\infty$.

As in the certainty equivalent approach of Weitzman (1998), the consensus discount factors obtained through our equilibrium approach are averages of the individual discount factors proposed by the agents. However, except in the case of homogeneous pure time preference rates, i.e. $\rho_i = \rho$ for all i, our expressions for the rates are different from those of Weitzman (1998). There is a bias towards the more impatient agents in the consensus equilibrium discount rates. A possible interpretation is as follows. When considering its intertemporal rate of substitution, the group must weigh more the agents with a higher shadow price of the intertemporal budget constraint, i.e. the more impatient members of the group.

As far as asymptotic properties are concerned, we obtain, as in Jouini et al. (2009) that the relevant rate in the long run is given by the lowest individual discount rate. This rate corresponds to the discount rate of the most patient agent (lowest ρ_i) when there is no beliefs heterogeneity, or to the most pessimistic agent (lowest μ_i) when there is no pure time preference rate heterogeneity and all the agents have the same volatility parameter or to the least confident agent (highest σ_i^2) when there is no pure time preference rate heterogeneity and all the agents have the same drift parameter. Moreover, the equilibrium approach leads to decreasing discount rates, not only utility discount rates, but also consumption discount rates. As in the certainty approach of Weitzman (1998), this leads to use lower discount rates for long term projects in a cost-benefit analysis.

In the case of homogeneous beliefs, Equation (2.3) involves the covariance between ρ_i and $\exp\left(-\rho_i t\right)$ as in Lengwiler (2005). Equation (2.3) also gives us the expression for the consensus utility discount rate $\rho \equiv \sum_{i=1}^{N} \frac{w_i \rho_i \exp(-\rho_i t)}{\sum_{j=1}^{N} w_j \rho_j \exp(-\rho^j t)} \rho_i$. Although of the same nature, this expression is different from the one obtained through the Pareto optimality of Gollier and Zeckhauser (2005) or the Benthamite approach of Nocetti et al. (2008). Indeed, our weights in the weighted averages of the ρ_i are given by the quantities $w_i \rho_i \exp\left(-\rho_i t\right)$ whereas they are given by $\lambda_i \exp\left(-\rho_i t\right)$ in Gollier and Zeckhauser (2005) or Nocetti et al. (2008), where the (λ_i) are Pareto weights chosen by the social planner.

2.2. Comparison with other formula for the socially efficient discount rate

The following proposition clarifies the relation between our socially efficient discount rate and the different expressions that have been provided in the literature.

Proposition 2.2. *1. The equilibrium discount rate is lower than the pure time preference weighted arithmetic average of the individual discount rates, i.e.*

$$R_t \leq \sum_{i=1}^{N} \frac{w_i \rho_i}{\sum_{j=1}^{N} w_j \rho_j} R^i \text{ and } r_t \leq \sum_{i=1}^{N} \frac{w_i \rho_i}{\sum_{j=1}^{N} w_j \rho_j} r^i. \tag{2.4}$$

2. If the tastes and beliefs characteristics ρ_i and $b_i \equiv \mu_i - \sigma_i^2$ are independent or if they are comonotonic, i.e. individuals with higher tastes characteristics ρ_i have higher beliefs characteristics b_i, then the equilibrium discount rate is higher than an average of the individual discount rates, i.e.

$$R_t \geq -\frac{1}{t} \ln \sum_{i=1}^{N} w_i \exp\left(-R^i t\right). \tag{2.5}$$

3. If the tastes and beliefs characteristics ρ_i and $b_i \equiv \mu_i - \sigma_i^2$ are independent, then the equilibrium marginal discount rate is higher than an average of the individual marginal discount rates, i.e.

$$r_t \geq \sum_{i=1}^{N} \frac{w_i \exp\left(-r^i t\right)}{\sum_{j=1}^{N} w_j \exp\left(-r^j t\right)} r^i. \tag{2.6}$$

Equation (2.4) means that, as expected, the discount rate to use is lower than the simple arithmetic average (with the same weights) of the individual discount rates. Moreover, Equations (2.5) and (2.6) imply that, when tastes and beliefs characteristics are independent, our discount rates are higher than those of Weitzman (2001). This is intuitive since our weights are given by the pure time preference rates ρ_i hence higher weights are granted to higher individual discount rates.

3. Specific Distributions and Dominance Properties

Let us now determine the equilibrium discount rate for specific distributions of the agents discount rates (R^i).

3.1. Gaussian and gamma distributions

We whall consider continuous sets of agents. It is easy to show that the expression of the discount rates remains the same in the setting with a continuum of agents[8]. The problem is that according to Equations (2.1) and (2.2), we need to make extra assumptions on the joint distribution of (ρ_i, R^i) in order to determine the discount rates R and r.

Consider first the case with homogeneous pure time preference rates $\rho_i = \rho$, and with a normal distribution $\mathcal{N}(m, v^2)$ on the beliefs parameters $b_i = \mu_i - \sigma_i^2$. The discount rates (R^i) then follow a normal distribution $\mathcal{N}(\rho + m, v^2)$ and we easily obtain that $R_t = \rho + m - \frac{v^2}{2} t$. Reinschmidt (2002) obtains a similar formula for the consensus *utility* discount rate when the individual *utility* discount rates follow a normal distribution.

[8] See Section 4 for a proof in a more general setting.

Suppose now that utility discount rates and beliefs are independently[9] and gamma distributed. We obtain the following result.

Proposition 3.1. *If pure time preference rates ρ_i and beliefs $b_i = \mu_i - \sigma_i^2$ are independently distributed[10] with $\rho_i \sim \gamma(\alpha_1, \beta_1)$ and $b_i \sim \gamma(\alpha_2, \beta_2)$, then*

1. $R_t = -\frac{\alpha_1+1}{t} \ln \frac{\beta_1}{\beta_1+t} - \frac{\alpha_2}{t} \ln \frac{\beta_2}{\beta_2+t}$ *and* $r_t = \frac{\alpha_1+1}{\beta_1+t} + \frac{\alpha_2}{\beta_2+t} = \frac{m_1^2+v_1^2}{m_1+tv_1^2} + \frac{m_2^2}{m_2+tv_2}$ *where* (m_1, v_1^2) *and* (m_2, v_2^2) *respectively denote the mean and variance of* (ρ_i) *and* (b_i).

2. *If* $\beta_1 = \beta_2$ *then* $R^i \sim \gamma(\alpha, \beta)$ *with* $\alpha = \alpha_1 + \alpha_2$, $R_t = R_t^W + \frac{1}{t} \ln\left(1 + \frac{t}{\beta}\right)$ *and* $r_t = \frac{m^2+v^2}{m+tv^2} = r_t^W + \frac{1}{\beta+t}$ *where* r_t^W *and* R_t^W *respectively denote the marginal discount rate and the discount rate obtained through the certainty equivalent approach of Weitzman and where* (m, v) *denote the mean and variance of* (R^i).

A decrease in the mean m_2 or an increase in the variance v_2^2 of the individual beliefs (b_i) decreases the marginal discount rate r_t (hence the discount rate R_t). The same result occurs with a decrease in the mean m_1 of the individual pure time preference rates (ρ_i). An increase in the variance v_1^2 of the individual pure time preference rates (ρ_i) decreases the marginal discount rate r_t for t large enough.

When beliefs and tastes are independent and follow gamma distributions with the same parameter β, the distribution of the individual discount rates R^i or r^i is a sufficient statistics for the equilibrium discount rate. As in Weitzman (2001), agents discount rates then follow a gamma distribution. As shown in the previous section, our equilibrium discount rates are higher than Weitzman (2001)'s discount rates but converge to the same value. A decrease in the mean m of the individual discount rates (R^i) decreases the marginal discount rate and an increase in the variance v^2 of the individual discount rates (R^i) decreases the marginal discount rate r_t for t large enough.

3.2. Calibration on Weitzman (2001)'s data

We now calibrate this model with two independent gamma distributions $\rho_i \sim \gamma(\alpha_1, \beta_1)$ and $b_i \sim \gamma(\alpha_2, \beta_2)$ on Weitzman (2001)'s data. We impose that $m_1 + m_2 = \bar{m}$ and $v_1^2 + v_2^2 = \bar{v}^2$ where $\bar{m}$ and $\bar{v}^2$ respectively denote the mean and the variance of the individual discount rates computed on Weitzman(2001)'s sample. We further impose that $\frac{m_1}{v_1} = \frac{m_2}{v_2}$ (same ratio between mean and standard deviation for both distributions), which leads to $\alpha_1 = \alpha_2$ and $\frac{\beta_1}{\beta_2} = \frac{m_2}{m_1} \equiv \lambda$, for some positive λ. Note that for $\lambda = 1$, we get $(\alpha_1, \beta_1, \alpha_2, \beta_2) = \left(\frac{m^2}{2v^2}, \frac{m}{v^2}, \frac{m^2}{2v^2}, \frac{m}{v^2}\right)$, which corresponds to the calibration in Weitzman (2001). We have then a family of stastistical models that contains Weitzman (2001)'s statistical model and we maximize the log-likelihood with respect to the parameter λ to choose the best calibration. Figure 1 represents the log-likelihood as a function of λ. The maximum is reached for $\lambda^* =$

[9] We focus on the case of independent distributions as a central case; explicit formulas may also be derived in the case of a given correlation between pure time preference rates and beliefs.

[10] Recall that the density function of a gamma distribution $\gamma(\alpha, \beta)$ is given by $\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x)$. Its mean m and its variance v^2 are respectively given by $m = \frac{\alpha}{\beta}$ and $\frac{\alpha}{\beta^2}$.

0.4116 hence $(\alpha_1, \beta_1, \alpha_2, \beta_2) = (1.043, 89.454, 1.043, 36.819)$ and $(m_1, v_1^2, m_2, v_2^2) = (1.16 \times 10^{-2}, 1.30 \times 10^{-4}, 2.83 \times 10^{-2}, 7.69 \times 10^{-4})$. To summarise, the best calibration corresponds to a gamma distribution on the individual pure time preference rates with an average time preference rate among agents equal to 1.16% and a median equal to 0.67% and a gamma distribution on the individual beliefs with an average belief parameter (about the growth of the economy) equal to 2.83% and a median equal to 2%. More precisely, the belief parameter $b = \mu - \sigma^2$ can be interpreted as a risk adjusted growth rate. The values we obtain are then reasonable values for both an average pure time preference rate and an average risk-adjusted growth rate. Stern report considers values for the pure time preference rate (utility discount rate) between 0.1 and 1.5 and values for the growth rate ranging from 0 per cent to 6 per cent. Arrow (1995) states that the pure time preference rate should be about 1% and surveying the evidence, the HM Treasury's Green Book (2003) suggests a long run growth rate of 2.1 per cent.

Figure 2 represents the marginal discount rate curve for the parameter λ that maximizes the log-likelihood ($\lambda^* = 0.4116$) and compares it to the discount rate curve of Weitzman (2001). Table 1 presents the corresponding recommended sliding-scale discount rates: Immediate Future about 5 per cent; Near Future about 4 percent; Medium Future about 3 percent; Distant Future about 1.5 per cent and Far-Distant Future about 0 per cent.

3.3. Dominance properties

In the specific setting of gamma distributions considered in Proposition 3.1, we have seen that the impact of a decrease in the mean or of an increase in the variance of the distribution of individual discount rates is towards a decrease of the socially efficient discount rates. The impact was the same for the average and for the marginal discount rates. We now analyze in a more general setting which shifts on the distribution of individual discount rates have a clear impact on the socially efficient discount rates. We consider First and Second Stochastic Dominance shifts, as defined in e.g. Rothschild and Stiglitz (1970). In order to obtain a clear impact on the socially efficient marginal discount rate r, we also consider Monotone Likelihood Ratio dominance (MLR) shifts. This concept of dominance has been studied by Landsberger and Meilijson (1990) and is defined as follows: a random variable Y dominates a random variable X, if X and Y have densities with respect to some dominating measure ν such that $f_X(x)f_Y(y) \leq f_X(y)f_Y(x)$ for all $y \leq x$ (roughly speaking, the ratio $\frac{f_Y}{f_X}$ is nondecreasing).

Proposition 3.2. *1. If all the agents have the same time preference rate ρ_i, then a FSD (resp. SSD) shift in the distribution of (R^i) increases the discount rate R_t for all horizons.*

2. If all the agents have the same time preference ρ_i, then a MLR shift in the distribution of the (r^i) increases the marginal discount rate r_t for all horizons.

3. If all the agents have the same beliefs, then a MLR shift in the distribution of the (R^i) increases the discount rate R_t for all horizons.

Proposition 3.2 makes it clear which concepts of dominance (corresponding to the notions of pessimism, doubt, patience, or heterogeneity) one should consider in order to have a clear impact on the discount rates. Roughly speaking, it means that a country where individuals are more pessimistic and/or exhibit more doubt about future growth and/or have lower pure time preference rates (more patient or more altruistic with respect to future generations) should apply a lower discount rate for cost-benefit analysis. More heterogeneity in agents beliefs about future growth rates also leads to lower discount rates.

Assume that all agents have the same time preference rate and suppose that in one population, say (A), we have three equally large groups with discount rates of 2%, 3% and 4%. In a second population (B), there are also three groups with the same anticipated growth rates but their proportion in the population is $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$. Population (B) is more pessimistic than population (A) (in the sense of the FSD) and the discount rate to apply is lower for (B). In a third population (C), there are three groups with anticipated growth rates 1%, 3% and 5% and their proportion in the population is $\frac{1}{10}$, $\frac{8}{10}$ and $\frac{1}{10}$. Populations (A) and (C) have the same average level of pessimism but population (C) is more heterogeneous (in the sense of the SSD) than population (A) and the discount rate to apply is lower for (C). Let us assume now that expert i anticipates a distribution $\mathcal{N}(\mu_i, \sigma_i^2)$ for the growth rate and provides forecasts with a 95% confidence interval. Let us assume that these intervals in population (A) are given by $[1.5, 2.5]$, $[2.5, 3.5]$ and $[3.5, 4.5]$ while in a fourth population (D) also with three equally large groups, these intervals are given by $[1;3]$, $[2;4]$ and $[3;5]$. There is more doubt in population (D) and the discount rate to apply is then lower for (D). The MLR (monotone likelihood ratio) dominance is stronger than the FSD dominance. Let us consider a population (F) with three groups that have the same discount rates as in (A) but with proportions in the population respectively equal to w_1, w_2 and w_3. The population (A) is more patient than (F) (in the sense of the MLR) if $w_3 < w_2 < w_1$. In this case, the discount rate to apply for cost-benefit analysis is lower for population (A).

4. Extensions

In this section, we examine two possible extensions: more general subjective and objective distributions for aggregate consumption and time dependent pure time preference rates.

4.1. General distributions for aggregate consumption

We first show that Proposition 2.1 remains valid in a very general complete markets Arrow-Debreu setting. Time can be continuous or discrete. We allow for a finite number or a continuum of agents. For this purpose, the set of agents is represented by a measured space $([0,1], \nu)$. Furthermore we do not need to assume specific individual distributions for aggregate consumption. It suffices to assume that agent i has a probability measure Q_t^i that represents the distribution of date$-t$ aggregate consumption from agent i point of view. As in previous sections, agent i has a pure time preference rate ρ_i, a share of total wealth w_i and a log-utility.

Proposition 4.1. *Let us consider a model with a measured space $([0,1], \nu)$ of log-utility agents that have pure time preference rates (ρ_i), wealth shares (w_i) and date$-t$ probability measures Q_t^i. We assume that all these probabilities are equivalent, i.e. the agents agree on* *the events of zero probability. The equilibrium discount rate is then given by*

$$R_t \equiv -\frac{1}{t}\ln\int_0^1 \frac{w_i\rho_i}{\int_0^1 w_j\rho_j d\nu(j)}\exp\left(-R_t^i t\right)d\nu(i) \tag{4.1}$$

where R_t^i is the equilibrium discount rate that would prevail if the economy was made of agent i only.

In such a general setting, the equilibrium discount rate is still a weighted average of the individual discount rates, and as previously, the weights are proportional to $w_i\rho_i$. The only difference with the setting of Equation (2.1) is the fact that the individual discount rates R_t^i may depend upon t.

4.2. Time dependent pure time preference rates

It is also easy to adapt our approach to the case with time-dependent pure time preference rates $(\rho_i(t))$. We then have $R_t^i = \frac{1}{t}\int_0^t \rho_i(s)ds + \mu_i - \sigma_i^2$ and $r_t^i = \rho_i(t) + \mu_i - \sigma_i^2$. We obtain the following expression for the socially efficient discount rate.

Proposition 4.2. *If agents have time-dependent positive pure time preference rates $(\rho_i(t))$, wealth shares (w_i) and date$-t$ distributions for aggregate consumption $\ln\mathcal{N}((\mu_i - \frac{1}{2}\sigma_i^2)t, \sigma_i^2 t)$, the equilibrium discount rates are given by*

$$R_t \equiv -\frac{1}{t}\ln\sum_{i=1}^N \frac{w_i\overline{\rho}_i}{\sum_{i=1}^N w_j\overline{\rho}_j}\exp\left(-R_t^i t\right) \tag{4.2}$$

$$r_t \equiv \sum_{i=1}^N \frac{w_i\overline{\rho}_i \exp\left(-\int_0^t r_s^i ds\right)}{\sum_{i=1}^N w_j\overline{\rho}_j \exp\left(-\int_0^t r_s^i ds\right)} r_t^i \tag{4.3}$$

for $R_t^i = \frac{1}{t}\int_0^t \rho_i(s)ds + \mu_i - \sigma_i^2$, $r_t^i = \rho_i(t) + \mu_i - \sigma_i^2$ and $\overline{\rho}_i = \left(\int_0^\infty \exp\left(-\int_0^t \rho_i(s)\,ds\right)dt\right)^{-1}$.

If the pure time preference rates $(\rho_i(t))$ are decreasing with t, then the discount rates R_t and r_t are also decreasing with t and we have

$$\lim_{t\to\infty} R_t = \lim_{t\to\infty} r_t = \inf_i \inf_t r_t^i = \inf_i \left(\mu_i - \sigma_i^2 + \lim_{t\to\infty}\rho_i(t)\right).$$

5. Conclusion

In this note, we propose a methodology in order to construct a consensus discount rate. We emphasize that our approach enables to deal with consumption discount rates and not only with utility discount rates (pure time preference rates).

We start with the recognition that divergence on what the discount rate should be is rooted in fundamental differences of opinion about inter-generational equity as well as about future growth. This enables us to translate the problem of aggregating discount rates into a problem of aggregating heterogeneous beliefs and time preference rates. We can then use the techniques of Jouini et al. (2009) in order to obtain the expression of the socially efficient discount rate. The equilibrium discount rate is a weighted average of the agents discount rates, in which more impatient agents are more heavily weighted; the equilibrium discount rate is decreasing and converges to the lowest individual discount rate, which does not necessarily correspond to the discount rate of the most 'patient' agent.

We show that the equilibrium discount rate is higher than Weitzman (1998)'s certainty equivalent discount rate for all horizon. More divergence of opinion about future growth rates among agents (in the form of second stochastic dominated shifts) leads to lower discount rates for all horizons. More doubt (larger confidence intervals) as well as more pessimism (in the form of first stochastic or monotone likelihood ratio dominated shifts) also leads to lower discount rates. This means that scientific uncertainty (more divergence of opinion or larger confidence intervals) has the same impact as pessimism and should lead to the use of low discount rates in cost-benefit anlysis.

We calibrate the model on Weitzman (2001)'s data. We show that the very wide spread of opinion on discount rates makes the effective equilibrium discount rate decline significantly over time from 5 per cent per annum for Immediate Future to 0 per cent per annum for Far-Distant Future. These very low discount rates impose to analyze carefully the costs and benefits of environnemental projects or activities even those that occur in far-distant future.

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Appendix

Derivation of the extended Ramsey equation

At the equilibrium, the date t state price density q_t is given by

$$q_t = \exp(-\rho t)u'(e_t)$$

and the discount rate is given by

$$R = -\frac{1}{t}\ln E\left[q_t\right] = \rho - \frac{1}{t}\ln E\left[e_t^{-\eta}\right].$$

The random variable $e_t^{-\eta}$ follows a log normal distribution with parameters $-\eta\left(\mu - \frac{1}{2}\sigma^2\right)t$ and $\sigma^2\eta^2 t$. We then have $\ln E\left[c^{-\frac{1}{\eta}}\right]$=$-t(\eta\mu - \frac{1}{2}\eta\left(1+\eta\right)\sigma^2)$ and

$$R = \rho + \eta\mu - \frac{1}{2}\eta\left(1+\eta\right)\sigma^2.$$

■

Proof of Equation 2.1 and of Proposition 2.1

We first prove that $A_t = \sum_{i=1}^{N}\gamma_i\exp\left(-R^i t\right) = \sum_{i=1}^{N}\gamma_i\exp\left(-r^i t\right)$, with $\gamma_i = \frac{w_i\rho_i}{\sum_{j=1}^{N}w_j\rho_j}$. The Euler conditions associated to the individual optimal consumption problems are given by

$$\frac{1}{\lambda_i}\exp(-\rho_i t)M_t^i\frac{1}{y_t^i} = q_t \qquad \text{for all } i$$

for some positive Lagrange multipliers (λ_i)

We have then

$$\frac{1}{\lambda_i}\exp(-\rho_i t)M_t^i\frac{1}{q_t} = y_t^i$$

and summing all these equations leads to

$$q_t = \sum_{i=1}^{N}\frac{1}{\lambda_i}\exp(-\rho_i t)M_t^i\frac{1}{e_t}.$$

Now, in our setting, $\exp\left(-r^i t\right) = \exp\left(-R^i t\right) = E\left[\exp(-\rho_i t)M_t^i\frac{1}{e_t}\right]$, hence

$$A_t = E\left[q_t\right] = \sum_{i=1}^{N}\frac{1}{\lambda_i}\exp\left(-r^i t\right) = \sum_{i=1}^{N}\frac{1}{\lambda_i}\exp\left(-R^i t\right).$$

It remains to determine the equilibrium weights $\frac{1}{\lambda_i}$. From the Euler and budget conditions we have

$$\int_0^\infty E^P\left[q_t y_t^i\right]dt = \frac{1}{\lambda_i\rho_i} = w_i\int_0^\infty E^P\left[q_t e_t\right]dt$$

which leads to

$$\frac{1}{\lambda_i} = \frac{\rho_i w_i}{\sum_{j=1}^{N}\rho_j w_j}.$$

We easily deduce Equations (2.1) , (2.2) as well as Equation (2.3) .

As far as the monotony of r is concerned, we have $r_t \equiv \sum_{i=1}^{N} \frac{w_i\rho_i \exp\left(-r^i t\right)}{\sum_{j=1}^{N} w_j\rho_j \exp(-r^j t)} r^i$, then $\frac{dr_t}{dt} = \frac{\left(\sum_{i=1}^{N} w_i\rho_i \exp\left(-r^i t\right) r^i\right)^2}{\left(\sum_{i=1}^{N} w_i\rho_i \exp(-r^i t)\right)^2} - \sum_{i=1}^{N} \frac{w_i\rho_i \exp\left(-r^i t\right)}{\sum_{j=1}^{N} w_j\rho_j \exp(-r^j t)} \left(r^i\right)^2$. Let us consider $P_{\rho \exp}$ the probability measure whose weights are proportional to $w_i\rho_i \exp\left(-r^i t\right)$. We have $\frac{dr_t}{dt} = E^{P_{\rho \exp}}\left[r\right]^2 - E^{P_{\rho \exp}}\left[r^2\right] \leq 0$. The marginal discount rate r_t decreases then with t.

As far as the asymptotic behavior of the marginal discount rate r is concerned, let $r^{i^*} \equiv \inf_i r^i$ and let j be such that $r^j \neq r^{i^*}$, then the relative weight of r^j in Equation (2.2) converges to zero and $r_t \to_{t\to\infty} r^{i^*}$. ■

Proof of Proposition 2.2

1. According to Proposition 2.1, we have

$$R_t \equiv -\frac{1}{t} \ln A_t$$

where A_t is an arithmetic average of the $\exp\left(-R^i t\right)$. Since the arithmetic average is larger than the geometric average, we have

$$\sum_{i=1}^{N} \frac{w_i\rho_i}{\sum_{j=1}^{N} w_j\rho_j} \exp\left(-R^i t\right) \geq \exp - \sum_{i=1}^{N} \frac{w_i\rho_i}{\sum_{j=1}^{N} w_j\rho_j} R^i t.$$

Hence,

$$R_t \leq \sum_{i=1}^{N} \frac{w_i\rho_i}{\sum_{j=1}^{N} w_j\rho_j} R^i.$$

We have

$$r_t \equiv \sum_{i=1}^{N} \frac{w_i\rho_i \exp\left(-r^i t\right)}{\sum_{j=1}^{N} w_j\rho_j \exp\left(-r^j t\right)} r^i = \frac{E^{P_\rho}\left[\exp\left(-rt\right) r\right]}{E^{P_\rho}\left[\exp\left(-rt\right)\right]}$$

where P_ρ has weights $\frac{w_i\rho_i}{\sum_{j=1}^{N} w_j\rho_j}$. Since $\exp\left(-r^i t\right)$ decreases with r^i, we have $E^{P_\rho}\left[\exp\left(-rt\right) r^i\right] \leq E^{P_\rho}\left[r\right] E^{P_\rho}\left[\exp\left(-rt\right)\right]$. Hence

$$r_t \leq E^{P_\rho}\left[r\right] = \sum_{i=1}^{N} \frac{w_i\rho_i}{\sum_{j=1}^{N} w_j\rho_j} r^i.$$

2. Let us denote by P_w the probability measure with weights w_i. Since ρ_i and $b_i \equiv \mu_i - \sigma_i^2$ are independent, we have

$$\exp\left(-R_t t\right) = \frac{E^{P_w}\left[\rho \exp\left(-\rho t\right) \exp\left(-bt\right)\right]}{E^{P_w}\left[\rho\right]} = \frac{E^{P_w}\left[\rho \exp\left(-\rho t\right)\right] E^{P_w}\left[\exp\left(-bt\right)\right]}{E^{P_w}\left[\rho\right]}.$$

Now, since ρ_i and $\exp\left(-\rho_i t\right)$ are anticomonotonic, we have

$$E^{P_w}\left[\rho \exp\left(-\rho t\right)\right] \leq E^{P_w}\left[\rho\right] E^{P_w}\left[\exp\left(-\rho t\right)\right],$$

which gives

$$\exp -R_t t \geq E^{P_w}\left[\exp\left(-\rho t\right)\right] E^{P_w}\left[\exp\left(-bt\right)\right] \geq E^{P_w}\left[\exp\left(-rt\right)\right].$$

3. We have

$$r_t = \frac{E^{P_{\exp}}\left[\rho^2\right] + E^{P_{\exp}}\left[\rho b\right]}{E^{P_{\exp}}\left[\rho\right]}.$$

where $P_{\exp}$ denotes the probability measure whose weights are proportional to $w_i \exp(-r^i t)$. We have then

$$\begin{aligned}
r_t &\geq \frac{E^{P_{\exp}}\left[\rho\right]^2 + E^{P_{\exp}}\left[\rho b\right]}{E^{P_{\exp}}\left[\rho\right]}, \\
&\geq E^{P_w}\left[\rho \exp\left(-\rho t\right)\right] \frac{E^{P_w}\left[\rho \exp\left(-\rho t\right)\right] E^{P_w}\left[\exp\left(-bt\right)\right] + E^{P_w}\left[\exp\left(-bt\right) b\right] E^{P_w}\left[\exp\left(-\rho t\right)\right]}{E^{P_w}\left[\exp\left(-\rho t\right)\right] E^{P_w}\left[\rho \exp\left(-rt\right)\right]}, \\
&\geq \frac{E^{P_w}\left[\left(\rho + b\right)\exp\left(-\left(\rho + b\right)t\right)\right]}{E^{P_w}\left[\exp\left(-\rho t\right)\right] E^{P_w}\left[\exp\left(-bt\right)\right]} \geq \sum_{i=1}^{N} \frac{w_i \exp\left(-r^i t\right)}{\sum_{j=1}^{N} w_j \exp\left(-r^j t\right)} r^i.
\end{aligned}$$

■

Proof of Proposition 3.1

If pure time preference rates ρ_i and beliefs $b_i = \mu_i - \sigma_i^2$ are independent and are distributed as follows $\rho_i \sim \gamma(\alpha_1, \beta_1)$ and $b_i \sim \gamma(\alpha_2, \beta_2)$, then

$$\begin{aligned}
A_t &= \frac{\beta_2^{\alpha_2}}{\Gamma\left(\alpha_2\right)} \frac{\beta_1^{\alpha_1}}{\Gamma\left(\alpha_1\right)} \frac{\int_0^\infty \int_0^\infty \rho \exp\left(-\left(\rho + b\right)t\right) \rho^{\alpha_1 - 1} \exp\left(-\beta_1 \rho\right) b^{\alpha_2 - 1} \exp\left(-\beta_2 b\right) d\rho db}{\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} \int_0^\infty \rho \rho^{\alpha_1 - 1} \exp\left(-\beta_1 \rho\right) d\rho}, \\
&= \left(\frac{\beta_1}{\beta_1 + t}\right)^{1+\alpha_1} \left(\frac{\beta_2}{\beta_2 + t}\right)^{\alpha_2}.
\end{aligned}$$

The results on the expression of R and r are then easily derived. ■

Proof of Proposition 3.2

1. Let us assume that all the agents have the same ρ_i, we have then

$$R_t \equiv -\frac{1}{t} \ln E^{P_w}\left[\exp\left(-Rt\right)\right]$$

where P_w is defined as in the proof of Proposition 2. For a given t, the function $R \to \exp\left(-Rt\right)$ is decreasing (and convex) and, by definition, a FSD (resp. SSD) shift in the distribution of (R^i) decreases the value of $E^{P_w}\left[\exp -Rt\right]$ and increases R_t.

2. We still assume that all the agents have the same ρ_i, we have then

$$r_t = \frac{E^{P_w}\left[r \exp\left(-rt\right)\right]}{E^{P_w}\left[\exp\left(-rt\right)\right]}.$$

Let us consider P_w^1 and P_w^2, two distributions such that $P_w^2 \succeq_{MLR} P_w^1$. By definition, the density $\phi = \frac{dP_w^2}{dP_w^1}$ is nondecreasing in r (in other words $i \to \phi^i$ and $i \to r^i$ are comonotonic).

We have then, $\frac{E^{P_w^2}[r\exp -rt]}{E^{P_w^2}[\exp(-rt)]} = \frac{E^{P_w^1}[\phi r\exp -rt]}{E^{P_w^1}[\phi\exp(-rt)]} = \frac{E^{Q_{\exp}}[\phi r]}{E^{Q_{\exp}}[\phi]}$ where $Q_{\exp}$ is defined by a density with respect to P_w^1 equal (up to a constant) to $\exp(-rt)$. Since ϕ is nondecreasing in r, we have

$$E^{Q_{\exp}}[\phi r] \geq E^{Q_{\exp}}[\phi]\, E^{Q_{\exp}}[r],$$

hence

$$\frac{E^{P_w^2}[r\exp -rt]}{E^{P_w^2}[\exp(-rt)]} \geq E^{Q_{\exp}}[r] \geq \frac{E^{P_w^1}[r\exp -rt]}{E^{P_w^1}[\exp -rt]}.$$

3. If we now assume that all the agents have the same belief, we have

$$R_t \equiv -\frac{1}{t}\ln\frac{E^{P_w}[\rho\exp -\rho t]}{E^{P_w}[\rho]}.$$

Let us consider P_w^1 and P_w^2, two distributions such that $P_w^2 \succeq_{MLR} P_w^1$. We have then, $\frac{E^{P_w^2}[\rho\exp -\rho t]}{E^{P_w^2}[\rho]} = \frac{E^{P_w^1}[\phi\rho\exp -\rho t]}{E^{P_w^1}[\phi\rho]} = \frac{E^{Q_\rho}[\phi\exp -\rho t]}{E^{Q_\rho}[\phi]}$ where $\phi = \frac{dP_w^2}{dP_w^1}$ and where Q_ρ is defined by a density with respect to P_w^1 equal (up to a constant) to ρ. Since ϕ is nondecreasing in r and then nonincreasing in $\exp -\rho t$, we have

$$E^{Q_\rho}[\phi\exp -\rho t] \leq E^{Q_\rho}[\phi]\, E^{Q_\rho}[\exp -\rho t],$$

hence

$$\frac{E^{P_w^2}[\rho\exp -\rho t]}{E^{P_w^2}[\rho]} \leq E^{Q_\rho}[\exp -\rho t] \leq \frac{E^{P_w^1}[\rho\exp -\rho t]}{E^{P_w^1}[\rho]}.$$

■

Proof of Proposition 4.1

Exactly along the same lines af the proof of Proposition **2.1**. ■

Proof of Proposition 4.2 It is easy to see that the formulas in the proof of Proposition **2.1** above have to be adapted as follows

$$\begin{aligned} q_t &= \frac{1}{\lambda_i}\exp(-\int_0^t \rho_i(s)ds)M_t^i\frac{1}{y_t^i} \\ \int_0^\infty E^P\left[q_t y_t^i\right]dt &= \int_0^\infty \frac{1}{\lambda_i}\exp(-\int_0^t \rho_i(s)ds)dt \end{aligned}$$

The same steps as above lead to

$$A_t = \sum_{i=1}^N \frac{\overline{\rho}_i w_i}{\sum_{j=1}^N \overline{\rho}_j w_j} A_t^i$$

with $\overline{\rho}_i = \left(\int_0^\infty \exp(-\int_0^t \rho_i(s)ds)dt\right)^{-1}$. ■

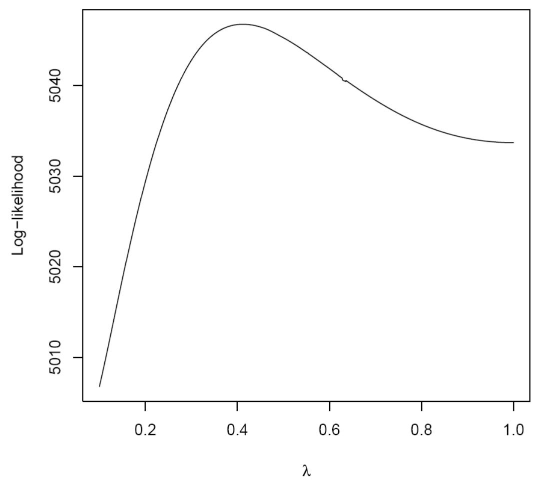

Figure 5.1: We calibrate a model with two independent gamma distributions (tastes and beliefs) on Weitzman (2001)'s data. We assume that the two distributions are homothetic (the first one is obtained from the second one through a change of variable $x \to \lambda x$ where λ is a given parameter) and we calibrate the model in order to fit the mean and the variance of the empirical distribution. We have then a family of statistical models that contains Weitzman (2001)'s statistical model (it corresponds to $\lambda = 1$) and we maximize the log-likelihood with respect to the parameter λ to choose the best calibration. We obtain $\lambda = 0.4116$.

Time period	Name	Numerical value	Approx. rate	Weitzman's num. value	Weitzman's appr. rate
Within years 1 to 5 hence	Immediate Future	4.99%	5%	3.89%	4%
Within years 6 to 25 hence	Near Future	4.23%	4%	3.22%	3%
Within years 26 to 75 hence	Medium Future	2.82%	3%	2.00%	2%
Within years 76 to 300 hence	Distant Future	1.50%	1.5%	0.97%	1%
Within years more than 300 hence	Far-Distant Future	0.16%	0%	0.08%	0%

Table 1 - Approximate recommended sliding-scale discount rates

We compare for different time periods the discount rates that result from our approach and those resulting from Weitzman (2001)'s approach. We first compute the rate that, if applied continuously from date 0 to the middle of the period would lead to the discount rate for that maturity. For next periods, we compute the rate that, if applied continuously from the beginning of the period to the middle of the period and compounded with the rates already computed for previous periods would lead to the discount rate for that maturity. Exact as well as approximate (recommended) results are provided for both approaches.

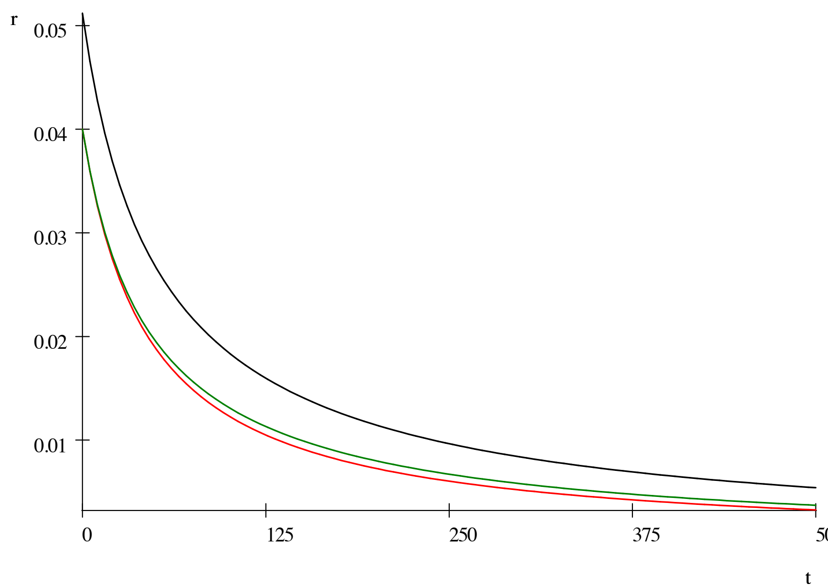

Figure 5.2: This figure represents the marginal discount rate curve $r_t = \frac{\alpha_1+1}{\beta_1+t} + \frac{\alpha_2}{\beta_2+t}$ obtained through our calibration (upper curve) and compares it to the discount rate curve $r_t = \frac{\alpha}{\beta+t}$ of Weitzman (2001) (lower curve). The intermediate curve represents the discount rates obtained through Weitzman's approach but with our calibration, $r_t = \frac{\alpha_1}{\beta_1+t} + \frac{\alpha_2}{\beta_2+t}$. It is clear that the difference between our discount rate curve and Weitzman (2001)'s curve mainly results from the fact that, contrarily to the certainty equivalent approach, more impatient agents are more heavily weighted in the equilibrium approach.